QUALITATIVE PROPERTIES OF POSITIVE SOLUTIONS TO NONLOCAL CRITICAL PROBLEMS INVOLVING THE HARDY-LERAY POTENTIAL

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ABSTRACT. We prove the existence, qualitative properties and asymptotic behavior of positive solutions to the doubly critical problem

$$(-\Delta)^s u = \vartheta \frac{u}{|x|^{2s}} + u^{2_s^*-1}, \quad u \in \dot{H}^s(\mathbb{R}^N).$$

The technique that we use to prove the existence is based on variational arguments. The qualitative properties are obtained by using of the moving plane method, in a nonlocal setting, on the whole $\mathbb{R}^N$ and by some comparison results.

Moreover, in order to find the asymptotic behavior of solutions, we use a representation result that allows to transform the original problem into a different nonlocal problem in a weighted fractional space.

1. Introduction

In this work we consider the doubly critical equation

$$(-\Delta)^s u = \vartheta \frac{u}{|x|^{2s}} + u^{2_s^*-1}, \quad u \in \dot{H}^s(\mathbb{R}^N), \tag{1.1}$$

with $N > 2s$, $0 < s < 1$, $2_s^* := \frac{2N}{N-2s}$ and $0 < \vartheta < \Lambda_{N,s}$, where $\Lambda_{N,s}$ is the sharp constant of the Hardy-Sobolev inequality, that is

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2s}}\, dx \le \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)|^2 \, d\xi, \qquad \text{for any } u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{1.2}$$

where $\mathcal{F}(u)$ denotes the Fourier transform of u. Moreover,

$$\Lambda_{N,s} = 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})},$$

where Γ is the so-called Gamma function. See [23], [6], [15] and [35].

We denote by $\mathcal{S}(\mathbb{R}^N)$ the class of all Schwartz functions in $\mathbb{R}^N$. Also, for any $f \in \mathcal{S}(\mathbb{R}^N)$, the fractional Laplacian of f will be denoted by $(-\Delta)^s f$, with $s \in (0,1)$. Namely, for any $x \in \mathbb{R}^N$,

$$(-\Delta)^s f := c_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^{N+2s}}\, dy,$$

where the constant $c_{N,s}$ is given by

$$c_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}}\, d\xi \right)^{-1} = 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}, \tag{1.3}$$

see [9, 16].

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Problem (1.1) can be considered as *doubly critical* due to the critical power in the semilinear term and the spectral anomaly of the Hardy potential. The fractional framework introduces nontrivial difficulties that have interest in itself. In this paper we want to show first the existence of solutions to problem (1.1) and then both qualitative properties of solutions and an asymptotic analysis of solutions at zero and at infinity.

First of all we need to define the natural functional framework for our problem, i.e. we consider the following Hilbert space:

Definition 1.1. *Let $0 < s < 1$. We define the homogeneous fractional Sobolev space of order s as*

$$\dot{H}^s(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) \,:\, |\xi|^s \mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^N)\},$$

namely the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2_{\dot{H}^s} := \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 \, d\xi. \tag{1.4}$$

By Plancherel's identity, we obtain an equivalent expression of the norm (1.4), as the following result states

Proposition 1.2. *Let $N \geq 1$ and $0 < s < 1$. Then for all $u \in \dot{H}^s(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 \, d\xi = c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy,$$

where $c_{N,s}$ is defined in (1.3)*.*

See for instance [15].

Remark 1.3. *According to Proposition 1.2 and using a density argument, the inequality in* (1.2) *can be reformulated in the following way:*

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2s}} \, dx \leq c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy, \qquad \text{for any } u \in \dot{H}^s(\mathbb{R}^N). \tag{1.5}$$

The notion of solutions to (1.1) that we consider in this paper is given in the following definition:

Definition 1.4. *We say that $u \in \dot{H}^s(\mathbb{R}^N)$, is a weak solution to* (1.1) *if, for every $\varphi \in \dot{H}^s(\mathbb{R}^N)$, we have:*

$$\frac{1}{2} c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy = \vartheta \int_{\mathbb{R}^N} \frac{u}{|x|^{2s}} \varphi \, dx + \int_{\mathbb{R}^N} u^{2_s^*-1} \varphi \, dx. \tag{1.6}$$

In the local case the problem

$$-\Delta u = A \frac{u}{|x|^2} + u^{2^*-1} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{1.7}$$

with $A \in [0, (N-2)^2/4)$ and $2^* := \frac{2N}{N-2}$, was studied in [34]. The author proves existence, uniqueness and qualitative properties of the solutions of problem (1.7) by a clever use of variational arguments and of the moving plane method. In particular it has been showed in [34] that the solution of (1.7) is unique (up to rescaling) and is given by

$$u_A(x) = \frac{(N(N-2)\eta_A^2)^{(N-2)/4}}{(|x|^{1-\eta_A}(1 + |x|^{2\eta_A}))^{(N-2)/2}}, \tag{1.8}$$

where

$$\eta_A := \left(1 - \frac{4A}{(N-2)^2}\right)^{1/2}.$$

In the nonlocal setting, in [10] the authors study the nonlocal problem

$$(-\Delta)^s u = u^{2^*_s - 1} \quad \text{in } \mathbb{R}^N. \tag{1.9}$$

They prove that every positive solution u of (1.9) is radially symmetric and radially decreasing about some point $x_0 \in \mathbb{R}^N$ and is given by

$$u(x) = c\left(\frac{t}{t^2 + |x - x_0|^2}\right)^{(N-2s)/2},$$

where c and t are positive constants. The authors use the moving plane method in an integral form and then they classify the solutions using that in fact problem (1.9) is equivalent to the integral equation

$$u(x) = \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2s}} u(y)^{\frac{N+2s}{N-2s}}\, dy. \tag{1.10}$$

Here we first show that problem (1.1) has a positive solution, according to Definition 1.4. Namely, we have the following:

Theorem 1.5. *Let $0 \leq \vartheta < \Lambda_{N,s}$. Then problem* (1.1) *has a positive solution.*

In order to prove Theorem 1.5, we deal with a constrained maximization problem (see in particular the forthcoming formula (2.1)) and then we use the Lagrange multipliers technique to get a solution to (1.1) and the main difficulty is to proof the compactness. Moreover, by local estimates we have that the positive solutions to (1.1), are strong solutions in $\mathbb{R}^N \setminus \{0\}$.

The second result that we prove in this paper is the radial symmetry of every solution to (1.1). That is, we have the following:

Theorem 1.6. *Assume that $0 \leq \vartheta < \Lambda_{N,s}$ and let u be a positive solution to* (1.1) *(in the sense of the Definition 1.4). Then u is radial and radially decreasing with respect to the origin. Namely there exists some strictly decreasing function $v : (0, +\infty) \to (0, +\infty)$ such that*

$$u(x) = v(r), \quad r = |x|.$$

In the local case such a result is generally proved exploiting the moving plane method which goes back to the seminal works of Alexandrov [4] and Serrin [32]. See in particular the celebrated papers [7, 21]. In the non local case we refer to [5, 10, 14, 26, 27].

Moreover, the presence of the Hardy potential in equation (1.1) makes difficult to use the technique developed in [10] where the authors exploit the equivalence of (1.1) to an integral equation like (1.10).

For this reason, in order to prove the radial symmetry of every solution to (1.1), we use an approach based on the moving plane method in all $\mathbb{R}^N$ for weak solutions of the equation.

Finally, we deal with the asymptotic behavior of solutions to (1.1) near the origin and at infinity. For this, we use a representation result by by Frank, Lieb and Seiringer, see [15]. In this way we are able to work with an equivalent equation of (1.1), see formula (4.6) and then

define a new nonlocal problem in some weighted fractional space. A similar procedure has been developed in [2] and [3] in order to solve some different elliptic and parabolic problems.

In particular, we first provide some regularity results (see Subsections 4.2 and 4.3). Then, the asymptotic analysis is contained in the following:

Theorem 1.7. *Let $u \in \dot{H}^s(\mathbb{R}^N)$ be a solution to (1.1). Then there exist two positive constants c and C such that*

$$\frac{c}{\left(|x|^{1-\eta_\theta}(1+|x|^{2\eta_\theta})\right)^{\frac{N-2s}{2}}} \le u(x) \le \frac{C}{\left(|x|^{1-\eta_\theta}(1+|x|^{2\eta_\theta})\right)^{\frac{N-2s}{2}}}, \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{1.11}$$

where

$$\eta_\theta = 1 - \frac{2\alpha_\theta}{N-2s} \tag{1.12}$$

and $\alpha_\theta \in (0, (N-2s)/2)$ is a suitable parameter whose explicit value will be determined as the unique solution to equation (4.3).

We point out the similarities between formulas (1.8) and (1.11). Indeed, the parameter η_θ in Theorem 1.7 plays the same role as the parameter η_A in the classical problem (1.7) (i.e. when $s = 1$ and the fractional Laplacian boils down to the classical Laplacian) both for the behavior near 0 and at infinity of the solution.

The paper is organized as follows: in Section 2 we show the existence of at least a solution to (1.1) and we prove Theorem 1.5. In Section 3 we study the qualitative properties of solutions to (1.1). We provide some maximum/comparison principle and we perform the moving plane method in order to get the radial symmetry of the solutions and prove then Theorem 1.6. Finally in Section 4 we investigate the behavior of solutions to (1.1) and we prove Theorem 1.7.

Notation. Generic fixed and numerical constants will be denoted by C (with subscript in some case) and they will be allowed to vary within a single line or formula.

2. Existence: the maximization problem and proof of Theorem 1.5

To prove the existence of a solution to (1.1) we consider the following maximization problem

$$S(\vartheta) := \sup_{u \in \dot{H}^s(\mathbb{R}^N)\setminus\{0\}} Q(u), \tag{2.1}$$

where

$$Q(u) := \frac{\displaystyle\int_{\mathbb{R}^N} |u|^{2^*_s}\,dx}{\left(\displaystyle\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \vartheta\int_{\mathbb{R}^N}\frac{u^2}{|x|^{2s}}\,dx\right)^{\frac{2^*_s}{2}}}.$$

Let us define the continuous bilinear form $\mathcal{L} : \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \to \mathbb{R}$ as

$$\mathcal{L}(u,v) := \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \vartheta\int_{\mathbb{R}^N}\frac{uv}{|x|^{2s}}\,dx \tag{2.2}$$

and the quadratic form $\tilde{\mathcal{L}} : \dot{H}^s(\mathbb{R}^N) \to \mathbb{R}$ as

$$\tilde{\mathcal{L}}(u) := \mathcal{L}(u,u) = \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \vartheta\int_{\mathbb{R}^N}\frac{u^2}{|x|^{2s}}\,dx. \tag{2.3}$$

We point out that, for $0 \le \vartheta < \Lambda_{N,s}$, a direct application of Hardy inequality (see (1.5)) shows that $(\tilde{\mathcal{L}}(u))^{1/2}$ is an equivalent norm in $\dot{H}(\mathbb{R}^N)$. We readily note that, by Hardy inequality, we have that $S(\vartheta) < +\infty$.

Moreover it easy to see that, if u is a maximum of the problem (2.1), then all the rescaled functions of u of the form

$$\sigma^{-\frac{N-2s}{2}}u\Big(\frac{\cdot}{\sigma}\Big), \quad \sigma > 0 \tag{2.4}$$

are also solutions to the maximization problem (2.1).

To get the existence, we take advantage of some improved Sobolev inequalities, see [29, Theorem 1.1]. In particular, in the proof of Theorem 1.5 we will use the fact that, for a function $u \in \dot{H}^s(\mathbb{R}^N)$, we have that

$$\|u\|_{L^{2^*_s}} \le C\|u\|^{\theta}_{\dot{H}^s}\|u\|^{1-\theta}_{\mathcal{L}^{2,N-2s}}, \tag{2.5}$$

where $2/2^*_s \le \theta < 1$ and $\|\cdot\|_{\mathcal{L}^{2,N-2s}}$ denotes the norm in the Morrey space $\mathcal{L}^{2,N-2s}$, that is

$$\|u\|^2_{\mathcal{L}^{2,N-2s}} := \sup_{R>0;\,x\in\mathbb{R}^N}\frac{R^{N-2s}}{|B_R(x)|}\int_{B_R(x)}|u|^2\,dz. \tag{2.6}$$

Proof of Theorem 1.5. We start finding a maximizing sequence $\{v_n\}$ that consist of radial and radially decreasing functions, i.e. $v_n(x) = v_n(|x|)$ for any $x \in \mathbb{R}^N$. In fact, let us first consider a maximizing sequence $\{u_n\} \in \dot{H}^s(\mathbb{R}^N)$ for (2.1). Notice that it is not restrictive to assume that $u_n(x) \ge 0$ a.e. in $\mathbb{R}^N$ (if not take $|u_n(x)|$).

Define $v_n(x) := u_n^*(x)$, where by f^* we denote the decreasing rearrangement of a measurable function f (where f is such that all its positive level set have finite measure), namely

$$f^*(x) = \inf\{t > 0 \,:\, |\{y \in \mathbb{R}^N \,:\, u(y) > t\}| \le \omega_N|x|^N\},$$

where ω_N is the volume of the standard unit N-sphere. By using a Polya-Szegö type inequality, see [30], we have that

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \ge \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u^*(x)-u^*(y)|^2}{|x-y|^{N+2s}}\,dx\,dy$$

and, by rearrangement properties, we also have that

$$\int_{\mathbb{R}^N}|u|^{2^*_s}\,dx = \int_{\mathbb{R}^N}|u^*|^{2^*_s}\,dx, \quad \int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\,dx \le \int_{\mathbb{R}^N}\frac{(u^*)^2}{|x|^2}\,dx.$$

Then the sequence $\{v_n\}$ is also a maximizing sequence (of radial and decreasing functions) for (2.1). Thanks to the homogeneity of (2.3), we do assume $\tilde{\mathcal{L}}(v_n) = 1$ for all n and

$$\int_{\mathbb{R}^N}|v_n|^{2^*_s}\,dx \to S(\vartheta) > 0, \quad \text{as} \;\; n \to +\infty. \tag{2.7}$$

Now we are going to show that a suitable rescaled sequence, that we will denote by $\hat{v}_n$, converges to a nontrivial weak limit, that is

$$\hat{v}_n \rightharpoonup v \not\equiv 0 \quad \text{in } \dot{H}^s(\mathbb{R}^N) \qquad \text{as } n \to +\infty. \tag{2.8}$$

To prove this, we first observe that, from (2.5) and (2.7), we have that

$$\|v_n\|_{\mathcal{L}^{2,N-2s}} \geq C > 0,$$

for some C independent of n.

By (2.6) and the fact that $\tilde{\mathcal{L}}(v_n) = 1$, for any $n \in \mathbb{N}$, we get the existence of $R_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\frac{1}{R_n^{2s}} \int_{B_{R_n}(x_n)} |v_n(z)|^2 \, dz \geq C > 0, \tag{2.9}$$

for some new positive constant C that does not depend on n.

Now we define the sequence $\hat{v}_n$ (of symmetric, radial decreasing functions) as

$$\hat{v}_n(x) := R_n^{\frac{N-2s}{2}} v_n(R_n x).$$

Notice that, by (2.4),

$$\int_{\mathbb{R}^N} |\hat{v}_n|^{2_s^*} \, dx = \int_{\mathbb{R}^N} |v_n|^{2_s^*} \, dx \to S(\vartheta) > 0, \quad \text{as} \quad n \to +\infty. \tag{2.10}$$

Moreover, using again the scaling invariance (2.4), we still have

$$\tilde{\mathcal{L}}(\hat{v}_n) = 1 \tag{2.11}$$

and

$$\|\hat{v}_n\|_{\dot{H}^s} \leq C.$$

Then, there exists $v \in \dot{H}^s(\mathbb{R}^N)$ such that, up to subsequences, $\hat{v}_n \rightharpoonup v$ in $\dot{H}^s(\mathbb{R}^N)$ as $n \to +\infty$.

Hence, to finish the proof of (2.8), it remains to show that

$$v \not\equiv 0. \tag{2.12}$$

To do this, we change variable in (2.9) and we obtain that

$$\int_{B_1(\hat{x}_n)} |\hat{v}_n(x)|^2 \, dx \geq C > 0, \tag{2.13}$$

where $\hat{x}_n := x_n / R_n$. Now we deal with two cases separately.

(i) Let us suppose that, up to subsequence, the sequence of points $\hat{x}_n \to \infty$. From (2.13) we infer that for every n there exists a set A_n of positive Lebesgue measure, such that $\hat{v}_n(x) > 0$, if $x \in A_n$. Since the sequence $\hat{v}_n$ consists of radial and radially decreasing functions, we have for all n that

$$\int_{B_2(0)} |\hat{v}_n(x)|^2 \, dx \geq \int_{B_1(0)} |\hat{v}_n(x + \hat{x}_n)|^2 \, dx = \int_{B_1(\hat{x}_n)} |\hat{v}_n(x)|^2 \, dx \geq C > 0.$$

Therefore, since the embedding of $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^p_{loc}$, $1 \leq p < 2_s^*$ is compact, see [16, Corollary 7.2], we deduce that

$$\int_{B_2(0)} |v(x)|^2 \geq C$$

and thus $v \not\equiv 0$. This shows (2.12) in this case.

(ii) Let us suppose up to subsequence $\hat{x}_n \to x_0$. Then, in this case, we fix a compact set $\mathcal{K}$ sufficiently large such that $B_2(x_0) \subset \mathcal{K}$. Making again use of (2.13), we have that, for n large enough,

$$\int_{\mathcal{K}} |\hat{v}_n(x)|^2\, dx \geq \int_{B_1(\hat{x}_n)} |\hat{v}_n(x)|^2\, dx \geq C > 0.$$

Therefore, the L^2 strong convergence on $\mathcal{K}$ implies that

$$\int_{\mathcal{K}} |v(x)|^2 \geq C$$

and then $v \not\equiv 0$. This concludes the proof of (2.12) also in this case.

Having finished the proof of (2.12), we obtain (2.8).

Now, since $\{\hat{v}_n\}$ is a maximizing sequence, we can show that actually

$$\hat{v}_n \to v \text{ strongly in } \dot{H}^s(\mathbb{R}^N). \tag{2.14}$$

In order to prove this, we observe that, recalling (2.2) and (2.3),

$$\tilde{\mathcal{L}}(v) + \tilde{\mathcal{L}}(\hat{v}_n - v) = 1 + o(1), \tag{2.15}$$

where $o(1)$ denotes a quantity that tends to zero as $n \to +\infty$. Indeed, we have that

$$\tilde{\mathcal{L}}(\hat{v}_n) = \tilde{\mathcal{L}}(v + \hat{v}_n - v) = \tilde{\mathcal{L}}(v) + \tilde{\mathcal{L}}(\hat{v}_n - v) + 2\mathcal{L}(v, \hat{v}_n - v).$$

Moreover, by (2.8)

$$\mathcal{L}(v, \hat{v}_n - v) \to 0 \qquad \text{as } n \to +\infty.$$

The last two formulas and (2.11) imply (2.15).

Furthermore, since $\hat{v}_n \to v$ a.e. (due to the compact embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^p_{loc}$, $1 \leq p < 2^*_s$, see [16, Corollary 7.2]), by (2.10) and Brezis-Lieb result [8] we have the following

$$\begin{aligned} S(\vartheta) = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\hat{v}_n|^{2^*_s}\, dx &= \lim_{n\to+\infty} \left(\int_{\mathbb{R}^N} |v|^{2^*_s}\, dx + \int_{\mathbb{R}^N} |\hat{v}_n - v|^{2^*_s}\, dx \right) \\ &\leq S(\vartheta)\Big(\tilde{\mathcal{L}}(v)\Big)^{\frac{2^*_s}{2}} + S(\vartheta)\Big(\lim_{n\to+\infty} \tilde{\mathcal{L}}(\hat{v}_n - v)\Big)^{\frac{2^*_s}{2}} \\ &\leq S(\vartheta)\Big(\tilde{\mathcal{L}}(v) + \lim_{n\to+\infty} \tilde{\mathcal{L}}(\hat{v}_n - v)\Big)^{\frac{2^*_s}{2}} \leq S(\vartheta), \end{aligned}$$

where we used (2.15) in the last line. Therefore, all the inequalities above have to be equalities. Moreover, since $v \not\equiv 0$, we infer that $\tilde{\mathcal{L}}(v) = 1$ and $\lim_{n\to+\infty} \tilde{\mathcal{L}}(\hat{v}_n - v) = 0$, i.e. $\hat{v}_n \to v$ strongly in $\dot{H}^s(\mathbb{R}^N)$ (recall that $(\tilde{\mathcal{L}}(\cdot))^{1/2}$ is an equivalent norm in $\dot{H}^s(\mathbb{R}^N)$). This shows (2.14).

As a consequence of (2.14) and using the fractional Sobolev embedding, we have that $\hat{v}_n \to v$ strongly in $L^{2^*_s}(\mathbb{R}^N)$ as well. Also, v turns to be a maximum for (2.1).

It is now standard by Lagrange multiplier Theorem to get the existence of a solution to (1.1), and so the proof of Theorem 1.5 is finished. □

3. Symmetry of Solutions and proof of Theorem 1.6

In this section we show that all the solution of (1.1) are radial and radially decreasing with respect to the origin, as stated in Theorem 1.6.

3.1. **Comparison principles.** In this subsection, relying on some results of Silvestre (see Section 2 in [33]), we provide maximum/comparison principle that will be useful in the application of the moving plane method in the forthcoming Subection 3.2.

For this, we introduce some notations. Let

$$\Phi(x) := \frac{C}{|x|^{N-2s}} \tag{3.1}$$

be the fundamental solution of $(-\Delta)^s$. We denote by Γ the regularization of Φ constructed in [33] (see Figure 2.1 there) and we set, for any $\tau > 1$ and for any $x \in \mathbb{R}^N$,

$$\Gamma_\tau(x) := \frac{\Gamma\left(\frac{x}{\tau}\right)}{\tau^{N-2s}} \tag{3.2}$$

and

$$\gamma_\tau(x) := (-\Delta)^s \Gamma_\tau(x). \tag{3.3}$$

The function $\Gamma_\tau \in C^{1,1}(\mathbb{R}^N)$ coincides with Φ outside $B_\tau(0)$ and it is a paraboloid inside $B_\tau(0)$ (see Section 2.2 in [33]), that is

$$\Gamma_\tau(x) = \Phi(x) \text{ in } \mathbb{R}^N \setminus B_\tau(0) \tag{3.4}$$

$$\text{and} \qquad \Gamma_\tau(x) \leq \Phi(x) \text{ in } B_\tau(0). \tag{3.5}$$

Moreover, the function γ_τ is strictly positive, thanks to Proposition 2.12 in [33].

Furthermore, given a function $\omega \in \dot{H}^s(\mathbb{R}^N) \cap C(\overline{\Omega})$, we say that ω satisfies

$$(-\Delta)^s \omega \geq 0 \qquad \text{in} \quad \Omega,$$

if

$$\frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\omega(x) - \omega(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy \geq 0,$$

for any nonnegative test function φ with compact support in Ω.

We denote a point $x \in \mathbb{R}^N$ as $x = (x_1, x_2, \cdots, x_N)$ and, for any $\lambda \in \mathbb{R}$, we set

$$\Sigma_\lambda := \left\{ x \in \mathbb{R}^N : x_1 < \lambda \right\} \tag{3.6}$$

and

$$T_\lambda := \left\{ x \in \mathbb{R}^N : x_1 = \lambda \right\}. \tag{3.7}$$

For any $\lambda \in \mathbb{R}$, we also set

$$x_\lambda := (2\lambda - x_1, x_2, \cdots, x_N). \tag{3.8}$$

With these definitions we can prove the following maximum principle:

Proposition 3.1. *Let $\lambda \in \mathbb{R}$ and $\Omega \Subset \Sigma_\lambda$ be a bounded open set. Let also $\omega \in \dot{H}^s(\mathbb{R}^N) \cap C(\overline{\Omega})$ that satisfies*

$$(-\Delta)^s \omega \geq 0 \qquad \text{in} \quad \Omega\,.$$

Assume that ω is nonnegative in Σ_λ and odd with respect to the hyperplane T_λ.

Then, either $\omega \equiv 0$ in $\mathbb{R}^N$ or $\omega > 0$ in Ω.

Remark 3.2. *The proof that we are going to exploit is the one in* [33] *(see Proposition 2.17). Some changes are needed since in our case we do not assume that ω is nonnegative in the whole space but, for future use, we assume that ω is odd with respect to the hyperplane T_λ. We could say that, in some sense, we agree that ω can have a negative part, but the latter has to be not to large.*

Proof of Proposition 3.1. If $\omega > 0$ in Ω, then the claim is true. Therefore, suppose that there exists a point $x_0 \in \Omega$ such that $\omega(x_0) = 0$. Hence, by [33, Proposition 2.15], we have that, for $\tau < \mathrm{dist}(x_0, \partial\Omega)$,

$$0 = \omega(x_0) \geq \int_{\mathbb{R}^N} \omega(x)\, \gamma_\tau(x - x_0)\, dx\,, \tag{3.9}$$

where γ_τ is defined in (3.3).

We claim that, for $\tau < \mathrm{dist}(x_0, \partial\Omega)$,

$$\int_{\mathbb{R}^N} \omega(x)\, \gamma_\tau(x - x_0)\, dx \geq 0\,. \tag{3.10}$$

For this, we notice that

$$\gamma_\tau(x - x_0) \geq \gamma_\tau(x_\lambda - x_0) > 0 \qquad \text{for any } x \in \Sigma_\lambda. \tag{3.11}$$

Indeed, if $x \in \Sigma_\lambda \setminus B_\tau(x_0)$, then

$$\begin{aligned} \gamma_\tau(x - x_0) &= \int_{\mathbb{R}^N} \frac{\Gamma_\tau(x - x_0) - \Gamma_\tau(y)}{|x - x_0 - y|^{N+2s}}\, dy \\ &= \int_{\mathbb{R}^N} \frac{\Phi(x - x_0) - \Phi(y)}{|x - x_0 - y|^{N+2s}}\, dy \,+\, \int_{\mathbb{R}^N} \frac{\Phi(y) - \Gamma_\tau(y)}{|x - x_0 - y|^{N+2s}}\, dy \\ &= \int_{B_\tau(x_0)} \frac{\Phi(y - x_0) - \Gamma_\tau(y - x_0)}{|x - y|^{N+2s}}\, dy\,, \end{aligned} \tag{3.12}$$

where in the last step we have used the fact that Φ is the fundamental solution to $(-\Delta)^s$ and (3.4) (all the integrals have to be intended in the principal value sense). Similarly, one has (again in the principal value sense)

$$\gamma_\tau(x_\lambda - x_0) = \int_{B_\tau(x_0)} \frac{\Phi(y - x_0) - \Gamma_\tau(y - x_0)}{|x_\lambda - y|^{N+2s}}\, dy. \tag{3.13}$$

Since $|x - y| \leq |x_\lambda - y|$ if $x \in \Sigma_\lambda \setminus B_\tau(x_0)$ and $y \in B_\tau(x_0)$, from (3.12) and (3.13) we obtain that

$$\gamma_\tau(x - x_0) \geq \gamma_\tau(x_\lambda - x_0) \text{ for any } x \in \Sigma_\lambda \setminus B_\tau(x_0). \tag{3.14}$$

Let now $x \in B_\tau(x_0)$. In [33] it has been proved that, for τ small,

$$\gamma_\tau(x' - x_0) \leq \frac{c\,\tau^{2s}}{|x' - x_0|^{N+2s}} \quad \text{for } |x' - x_0| \geq \frac{\mathrm{dist}(x_0, T_\lambda)}{2}, \tag{3.15}$$

where $c > 0$. Notice that, if $x \in B_\tau(x_0)$, then (3.15) holds for $x' := x_\lambda$. In particular, for τ small,

$$\gamma_\tau(x_\lambda - x_0) \leq C,$$

for some positive constant C. Moreover

$$\gamma_\tau(x - x_0) = \frac{1}{\tau^N} \gamma_1 \left(\frac{x - x_0}{\tau} \right) .$$

Now, we choose τ sufficiently small such that

$$\frac{1}{\tau^N}\gamma_1\left(\frac{x-x_0}{\tau}\right)\geq C,$$

and this implies that

$$\gamma_\tau(x-x_0)\geq\gamma_\tau(x_\lambda-x_0)\geq 0 \qquad \text{for} \quad \text{for any } x\in B_\tau(x_0)\,. \tag{3.16}$$

Putting together (3.14) and (3.16) we obtain (3.11).

Now, in order to prove (3.10), we write

$$\int_{\mathbb{R}^N}\omega(x)\,\gamma_\tau(x-x_0)\,dx=\int_{\Sigma_\lambda}\omega(x)\,\gamma_\tau(x-x_0)\,dx+\int_{\mathbb{R}^N\setminus\Sigma_\lambda}\omega(x)\,\gamma_\tau(x-x_0)\,dx.$$

Therefore, (3.10) is equivalent to show that

$$\int_{\Sigma_\lambda}\omega(x)\,\gamma_\tau(x-x_0)\,dx\geq-\int_{\mathbb{R}^N\setminus\Sigma_\lambda}\omega(x)\,\gamma_\tau(x-x_0)\,dx. \tag{3.17}$$

For this, we recall that $\omega\geq 0$ in Σ_λ and it is odd with respect to T_λ, and so, using (3.11) and the change of variable $y=x_\lambda$, we have

$$\begin{aligned}\int_{\Sigma_\lambda}\omega(x)\,\gamma_\tau(x-x_0)\,dx &\geq \int_{\Sigma_\lambda}\omega(x)\,\gamma_\tau(x_\lambda-x_0)\,dx\\ &= -\int_{\Sigma_\lambda}\omega(x_\lambda)\,\gamma_\tau(x_\lambda-x_0)\,dx\\ &= -\int_{\mathbb{R}^N\setminus\Sigma_\lambda}\omega(y)\,\gamma_\tau(y-x_0)\,dx.\end{aligned}$$

This implies (3.17), and therefore (3.10).

As a consequence, from (3.9) and (3.10), we have

$$\int_{\mathbb{R}^N}\omega(x)\,\gamma_\tau(x-x_0)\,dx\;=\;0\,.$$

Since γ_τ is strictly positive, this implies that $\omega=0$ a.e. in $\mathbb{R}^N$ and concludes the proof. $\square$

3.2. **Radial symmetry of the solutions.** In this section we prove that every solution $u\in\dot H^s(\mathbb{R}^N)$ to (1.1) is actually symmetric and monotone decreasing around the origin. The proof will be carried out exploiting the moving plane method. For the case of bounded domain in the nonlocal case we refer to [5, 14, 26, 27]. To do this, we start considering without lost of generality the x_1-direction.

For any $\lambda\in\mathbb{R}$, we recall the definitions of Σ_λ and T_λ given in (3.6) and (3.7), respectively. Moreover, we also set, for $x=(x_1,\dots,x_N)\in\mathbb{R}^N$ and x_λ defined in (3.8),

$$u_\lambda(x):=u(x_\lambda).$$

A point in $\mathbb{R}^N\times\mathbb{R}^N$ is denoted by (x,y) with $x,y\in\mathbb{R}^N$.

With these definitions, we have that, for any $\varphi \in \dot{H}^s(\mathbb{R}^N)$,

$$\begin{aligned}
& \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_\lambda(x) - u_\lambda(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy \\
= \; & \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x_\lambda) - u(y_\lambda))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy \\
= \; & \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(t) - u(z))(\varphi(t_\lambda) - \varphi(z_\lambda))}{|t_\lambda - z_\lambda|^{N+2s}} \, dt \, dz \\
= \; & \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(t) - u(z))(\varphi(t_\lambda) - \varphi(z_\lambda))}{|t - z|^{N+2s}} \, dt \, dz \\
= \; & \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(t) - u(z))(\varphi_\lambda(t) - \varphi_\lambda(z))}{|t - z|^{N+2s}} \, dt \, dz \\
= \; & \vartheta \int_{\mathbb{R}^N} \frac{u(t)}{|t|^{2s}} \varphi_\lambda(t) \, dt + \int_{\mathbb{R}^N} u^{2_s^*-1}(t) \varphi_\lambda(t) \, dt \\
= \; & \vartheta \int_{\mathbb{R}^N} \frac{u_\lambda(x)}{|x_\lambda|^{2s}} \varphi(x) \, dx + \int_{\mathbb{R}^N} u_\lambda^{2_s^*-1}(x) \varphi(x) \, dx,
\end{aligned}$$

where the changes of variables $t = x_\lambda$ and $z = y_\lambda$ and (1.6) were used. Notice that, if $\varphi \in \dot{H}^s(\mathbb{R}^N)$, then $\varphi_\lambda \in \dot{H}^s(\mathbb{R}^N)$ and so φ_λ can be used as a test function in (1.6).

As a consequence, u_λ is a weak solution to

$$(-\Delta)^s u_\lambda = \vartheta \frac{u_\lambda}{|x_\lambda|^{2s}} + u_\lambda^{2_s^*-1} \quad \text{in } \mathbb{R}^N. \tag{3.18}$$

Now we prove the following:

Lemma 3.3. *Let $0 \le \vartheta < \Lambda_{N,s}$ and let u be a positive solution to* (1.1)*, in the sense of Definition 1.4. Then*

$$\lim_{|x| \to 0} u(x) = +\infty. \tag{3.19}$$

Proof. By the weak Harnack inequality we have that $\inf_{B_2(0)} u > \delta > 0$. In particular,

$$(-\Delta)^s u(x) \ge \frac{\delta}{|x|^{2s}}, \quad x \in B_2(0).$$

Let now w be the solution to the problem

$$\begin{cases} (-\Delta)^s w(x) = \dfrac{\delta}{|x|^{2s}}, & x \in B_1(0) \\ w(x) = 0, & x \in \mathbb{R}^N \setminus B_1(0). \end{cases}$$

By comparison we obtain that $u \ge w$ in $B_1(0)$. Therefore, in order to prove Lemma 3.3, it is sufficient to prove that

$$\lim_{|x| \to 0} w(x) = +\infty. \tag{3.20}$$

For this, we define $\tilde{w}(x) := \frac{\delta}{|x|^{2s}} * \frac{C_{N,s}}{|x|^{N-2s}}$, where $\frac{C_{N,s}}{|x|^{N-2s}}$ is the fundamental solution of the fractional Laplace equation. So, for $|x| := 1/n$, we have that

$$
\begin{aligned}
\tilde{w}(x) &\geq C_{N,s}\,\delta \int_{B_\tau(0)} \frac{1}{|y|^{2s}|x-y|^{N-2s}}\,dy \\
&\geq \tilde{C} \int_{B_\tau(0)} \frac{1}{|y|^{2s}\left(|y|^{N-2s}+(1/n)^{N-2s}\right)}\,dy \to +\infty
\end{aligned}
$$

when $n \to +\infty$, that is, when $|x| \to 0$.

Also, we have that $w - \tilde{w}$ is harmonic in $B_1(0)$, and hence bounded in $B_{1/2}(0)$ (see e.g. [17, Proposition 4.1.1] and references therein). These considerations imply (3.20), as desired. $\square$

We are now in the position of completing the proof of Theorem 1.6.

Proof of Theorem 1.6. We take $\lambda < 0$ and we introduce the following function

$$
w_\lambda(x) := \begin{cases} (u-u_\lambda)^+(x), & \text{if } x \in \Sigma_\lambda, \\ (u-u_\lambda)^-(x), & \text{if } x \in \mathbb{R}^N \setminus \Sigma_\lambda, \end{cases} \tag{3.21}
$$

where $(u-u_\lambda)^+ := \max\{u-u_\lambda, 0\}$ and $(u-u_\lambda)^- := \min\{u-u_\lambda, 0\}$. We set

$$
\begin{aligned}
\mathcal{S}_\lambda &:= \mathit{supp}\ w_\lambda(x) \cap \Sigma_\lambda, & \mathcal{S}_\lambda^c &:= \Sigma_\lambda \setminus \mathcal{S}_\lambda, \\
\mathcal{D}_\lambda &:= \mathit{supp}\ w_\lambda(x) \cap \Big(\mathbb{R}^N \setminus \Sigma_\lambda\Big), & \mathcal{D}_\lambda^c &:= \Big(\mathbb{R}^N \setminus \Sigma_\lambda\Big) \setminus \mathcal{D}_\lambda\,.
\end{aligned} \tag{3.22}
$$

It is not difficult to see that

$$
\mathcal{D}_\lambda \text{ is the reflection of } \mathcal{S}_\lambda. \tag{3.23}
$$

Thanks to Lemma 3.3, we have that there exists $\rho = \rho(\lambda) > 0$ such that

$$
u < u_\lambda \quad \text{in} \quad B_\rho(0_\lambda) \subset \Sigma_\lambda,
$$

so that

$$
0 \notin \mathcal{S}_\lambda \cup \mathcal{D}_\lambda \qquad \text{and} \qquad 0_\lambda \notin \mathcal{S}_\lambda \cup \mathcal{D}_\lambda\,. \tag{3.24}
$$

We claim that

$$
w_\lambda \equiv 0 \text{ for } \lambda < 0 \text{ with } |\lambda| \text{ sufficiently large.} \tag{3.25}
$$

To prove this, we start noticing that the function w_λ defined in (3.21) belongs to $\dot{H}^s(\mathbb{R}^N)$ and so, recalling also (3.24), we can use it as test function in the weak formulations of (1.1) and (3.18). We have

(3.26)

$$
\begin{aligned}
\frac{c_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(w_\lambda(x)-w_\lambda(y))}{|x-y|^{N+2s}}\,dx\,dy &= \vartheta \int_{\mathbb{R}^N} \frac{u}{|x|^{2s}} w_\lambda\,dx + \int_{\mathbb{R}^N} u^{2_s^*-1} w_\lambda\,dx, \\
\frac{c_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\lambda(x)-u_\lambda(y))(w_\lambda(x)-w_\lambda(y))}{|x-y|^{N+2s}}\,dx\,dy &= \vartheta \int_{\mathbb{R}^N} \frac{u_\lambda}{|x_\lambda|^{2s}} w_\lambda\,dx + \int_{\mathbb{R}^N} u_\lambda^{2_s^*-1} w_\lambda\,dx.
\end{aligned}
$$

Subtracting the two equations in (3.26) we obtain

(3.27)
$$\begin{aligned} &\frac{1}{2}c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Big((u(x)-u_\lambda(x))-(u(y)-u_\lambda(y))\Big)\Big(w_\lambda(x)-w_\lambda(y)\Big)}{|x-y|^{N+2s}}\,dx\,dy\\ =\;\;&\vartheta\int_{\mathbb{R}^N}\left(\frac{u}{|x|^{2s}}-\frac{u_\lambda}{|x_\lambda|^{2s}}\right)w_\lambda\,dx+\int_{\mathbb{R}^N}(u^{2^*_s-1}-u_\lambda^{2^*_s-1})w_\lambda\,dx\\ \le\;\;&\vartheta\int_{\mathbb{R}^N}\frac{(u-u_\lambda)}{|x|^{2s}}w_\lambda\,dx+\int_{\mathbb{R}^N}(u^{2^*_s-1}-u_\lambda^{2^*_s-1})w_\lambda\,dx,\end{aligned}$$

since $|x|\ge|x_\lambda|$ and $w_\lambda\ge0$ in Σ_λ, and $|x|\le|x_\lambda|$ and $w_\lambda\le0$ outside Σ_λ. On the other hand, we have

(3.28)
$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Big((u(x)-u_\lambda(x))-(u(y)-u_\lambda(y))\Big)\Big(w_\lambda(x)-w_\lambda(y)\Big)}{|x-y|^{N+2s}}\,dx\,dy\\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Big(w_\lambda(x)-w_\lambda(y)\Big)^2}{|x-y|^{N+2s}}\,dx\,dy\\ &\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Big(\big(u(x)-u_\lambda(x)\big)-\big(u(y)-u_\lambda(y)\big)-\big(w_\lambda(x)-w_\lambda(y)\big)\Big)\Big(w_\lambda(x)-w_\lambda(y)\Big)}{|x-y|^{N+2s}}\,dx\,dy\\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Big(w_\lambda(x)-w_\lambda(y)\Big)^2}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy,\end{aligned}$$

where

$$\mathcal{G}(x,y)\,:=\,\Big(\big(u(x)-u_\lambda(x)\big)-\big(u(y)-u_\lambda(y)\big)-\big(w_\lambda(x)-w_\lambda(y)\big)\Big)\Big(w_\lambda(x)-w_\lambda(y)\Big)\,.$$

Now, we prove that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy\ge0\,. \tag{3.29}$$

To check this, we use the decomposition

$$\mathbb{R}^N\times\mathbb{R}^N\,=\,(\mathcal{S}_\lambda\cup\mathcal{S}_\lambda^c\cup\mathcal{D}_\lambda\cup\mathcal{D}_\lambda^c)\times(\mathcal{S}_\lambda\cup\mathcal{S}_\lambda^c\cup\mathcal{D}_\lambda\cup\mathcal{D}_\lambda^c)\,,$$

where $\mathcal{S}_\lambda$, $\mathcal{S}_\lambda^c$, $\mathcal{D}_\lambda$ and $\mathcal{D}_\lambda^c$ have been introduced in (3.22). By construction, it follows that

$$
\begin{aligned}
&\mathcal{G}(x,y)=\Big[-\big(u(x)-u_\lambda(x)\big)w_\lambda(y)\Big] && \text{in} \quad \big(\mathcal{S}_\lambda^c\times\mathcal{S}_\lambda\big),\\
&\mathcal{G}(x,y)=\Big[-\big(u(x)-u_\lambda(x)\big)w_\lambda(y)\Big] && \text{in} \quad \big(\mathcal{S}_\lambda^c\times\mathcal{D}_\lambda\big),\\
&\mathcal{G}(x,y)=\Big[-\big(u(y)-u_\lambda(y)\big)w_\lambda(x)\Big] && \text{in} \quad \big(\mathcal{S}_\lambda\times\mathcal{S}_\lambda^c\big),\\
&\mathcal{G}(x,y)=\Big[-\big(u(y)-u_\lambda(y)\big)w_\lambda(x)\Big] && \text{in} \quad \big(\mathcal{S}_\lambda\times\mathcal{D}_\lambda^c\big),\\
&\mathcal{G}(x,y)=\Big[-\big(u(x)-u_\lambda(x)\big)w_\lambda(y)\Big] && \text{in} \quad \big(\mathcal{D}_\lambda^c\times\mathcal{S}_\lambda\big),\\
&\mathcal{G}(x,y)=\Big[-\big(u(x)-u_\lambda(x)\big)w_\lambda(y)\Big] && \text{in} \quad \big(\mathcal{D}_\lambda^c\times\mathcal{D}_\lambda\big),\\
&\mathcal{G}(x,y)=\Big[-\big(u(y)-u_\lambda(y)\big)w_\lambda(x)\Big] && \text{in} \quad \big(\mathcal{D}_\lambda\times\mathcal{S}_\lambda^c\big),\\
&\mathcal{G}(x,y)=\Big[-\big(u(y)-u_\lambda(y)\big)w_\lambda(x)\Big] && \text{in} \quad \big(\mathcal{D}_\lambda\times\mathcal{D}_\lambda^c\big)\\
\text{and}\quad &\mathcal{G}(x,y)=0 && \text{elsewhere}\,.
\end{aligned}
$$

We have that

$$
\int_{\mathcal{S}_\lambda^c}\int_{\mathcal{S}_\lambda}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathcal{S}_\lambda^c}\int_{\mathcal{D}_\lambda}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy\geq 0\,. \tag{3.30}
$$

Indeed, notice that, if $x\in\mathcal{S}_\lambda^c$ and $y\in\mathcal{S}_\lambda$, then $\mathcal{G}(x,y)\geq 0$, and moreover

$$
\begin{aligned}
\mathcal{G}(x,y) &= -\big(u(x)-u_\lambda(x)\big)w_\lambda(y)\\
&= -\big(u(x)-u_\lambda(x)\big)\big(u(y)-u_\lambda(y)\big)\\
&= -\big(u(x)-u_\lambda(x)\big)\big(u_\lambda(y_\lambda)-u(y_\lambda)\big)\\
&= \big(u(x)-u_\lambda(x)\big)w_\lambda(y_\lambda)\\
&= -\mathcal{G}(x,y_\lambda).
\end{aligned}
$$

Also, we have that $|x-y|\leq|x-y_\lambda|$ in $\mathcal{S}_\lambda^c\times\mathcal{S}_\lambda$. Therefore, using also (3.23), we have

$$
\begin{aligned}
&\int_{\mathcal{S}_\lambda^c}\int_{\mathcal{S}_\lambda}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathcal{S}_\lambda^c}\int_{\mathcal{D}_\lambda}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad=\int_{\mathcal{S}_\lambda^c}\int_{\mathcal{S}_\lambda}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathcal{S}_\lambda^c}\int_{\mathcal{S}_\lambda}\frac{\mathcal{G}(x,y_\lambda)}{|x-y_\lambda|^{N+2s}}\,dx\,dy\\
&\quad=\int_{\mathcal{S}_\lambda^c}\int_{\mathcal{S}_\lambda}\mathcal{G}(x,y)\left[\frac{1}{|x-y|^{N+2s}}-\frac{1}{|x-y_\lambda|^{N+2s}}\right]\,dx\,dy\geq 0.
\end{aligned}
$$

which shows (3.30).

Similarly, one can prove that

$$
\int_{\mathcal{S}_\lambda}\int_{\mathcal{S}_\lambda^c}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathcal{S}_\lambda}\int_{\mathcal{D}_\lambda^c}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy\geq 0,
$$

$$
\int_{\mathcal{D}_\lambda^c}\int_{\mathcal{S}_\lambda}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathcal{D}_\lambda^c}\int_{\mathcal{D}_\lambda}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy\geq 0
$$

and

$$\int_{\mathcal{D}_\lambda}\int_{\mathcal{S}_\lambda^c}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathcal{D}_\lambda}\int_{\mathcal{D}_\lambda^c}\frac{\mathcal{G}(x,y)}{|x-y|^{N+2s}}\,dx\,dy\geq 0.$$

Collecting the estimates above we obtain (3.29).

Hence, from (3.28) and (3.29), we deduce that

$$\begin{aligned}&\frac{1}{2}c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Big((u(x)-u_\lambda(x))-(u(y)-u_\lambda(y))\Big)\Big(w_\lambda(x)-w_\lambda(y)\Big)}{|x-y|^{N+2s}}\,dx\,dy\\&\geq\frac{1}{2}c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_\lambda(x)-w_\lambda(y))^2}{|x-y|^{N+2s}}\,dx\,dy\,.\end{aligned}\tag{3.31}$$

Using this, (3.27) and the fact that $(u-u_\lambda)w_\lambda=w_\lambda^2$ in $\mathbb{R}^N$, we obtain

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_\lambda(x)-w_\lambda(y))^2}{|x-y|^{N+2s}}\,dx\,dy\leq\vartheta\int_{\mathbb{R}^N}\frac{w_\lambda^2}{|x|^{2s}}\,dx+\int_{\mathbb{R}^N}(u^{2_s^*-1}-u_\lambda^{2_s^*-1})w_\lambda\,dx.\tag{3.32}$$

For the first term in the right hand side of (3.32), we use Hardy inequality and we get

$$\vartheta\int_{\mathbb{R}^N}\frac{w_\lambda^2}{|x|^{2s}}\,dx\leq\frac{\vartheta}{\Lambda_{N,s}}\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_\lambda(x)-w_\lambda(y))^2}{|x-y|^{N+2s}}\,dx\,dy.\tag{3.33}$$

For the second term in the right hand side of (3.32), we first notice that, thanks to (3.23),

$$\int_{\mathcal{S}_\lambda}u^{2_s^*}\,dx=\int_{\mathcal{D}_\lambda}u_\lambda^{2_s^*}\,dx.$$

Moreover, $w_\lambda=0$ in $\mathcal{S}_\lambda^c\cup\mathcal{D}_\lambda^c$. Therefore, using also Lagrange Theorem and Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}(u^{2_s^*-1}-u_\lambda^{2_s^*-1})w_\lambda\,dx\\&=\int_{\mathcal{S}_\lambda}(u^{2_s^*-1}-u_\lambda^{2_s^*-1})w_\lambda\,dx+\int_{\mathcal{D}_\lambda}(u^{2_s^*-1}-u_\lambda^{2_s^*-1})w_\lambda\,dx\\&\leq C_1\int_{\mathcal{S}_\lambda}u^{2_s^*-2}\cdot w_\lambda^2\,dx+C_1\int_{\mathcal{D}_\lambda}u_\lambda^{2_s^*-2}\cdot w_\lambda^2\,dx\\&\leq C_1\left(\int_{\mathcal{S}_\lambda}u^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}}\left(\int_{\mathcal{S}_\lambda}w_\lambda^{2_s^*}\,dx\right)^{\frac{2}{2_s^*}}+C_1\left(\int_{\mathcal{D}_\lambda}u_\lambda^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}}\left(\int_{\mathcal{D}_\lambda}w_\lambda^{2_s^*}\,dx\right)^{\frac{2}{2_s^*}}\\&\leq C_2\left(\int_{\mathcal{S}_\lambda}u^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}}\left(\int_{\mathbb{R}^N}w_\lambda^{2_s^*}\,dx\right)^{\frac{2}{2_s^*}}\\&\leq C_3\left(\int_{\mathcal{S}_\lambda}u^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_\lambda(x)-w_\lambda(y))^2}{|x-y|^{N+2s}}\,dx\,dy,\end{aligned}\tag{3.34}$$

where we have also used the Sobolev embedding (see, for instance, Theorem 6.5 in [16]). Notice that the constants C_1, C_2 and C_3 are positive and independent of λ.

Collecting the inequalities in (3.32), (3.33) and (3.34), we obtain

(3.35)
$$\begin{aligned}&\left(\frac{c_{N,s}}{2}-\frac{\vartheta}{\Lambda_{N,s}}\frac{c_{N,s}}{2}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_\lambda(x)-w_\lambda(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ \leq\ & C_3\left(\int_{\mathcal{S}_\lambda}u^{2^*_s}\,dx\right)^{\frac{2^*_s-2}{2^*_s}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_\lambda(x)-w_\lambda(y))^2}{|x-y|^{N+2s}}\,dx\,dy.\end{aligned}$$

Since $u\in\dot H^s(\mathbb{R}^N)$ (and therefore in $L^{2^*_s}(\mathbb{R}^N)$), there exists $R>0$ such that for $\lambda<-R$ we have

$$C_3\left(\int_{\mathcal{S}_\lambda}u^{2^*_s}\,dx\right)^{\frac{2^*_s-2}{2^*_s}}\leq C_3\left(\int_{\Sigma_\lambda}u^{2^*_s}\,dx\right)^{\frac{2^*_s-2}{2^*_s}}\leq\frac12\left(\frac{c_{N,s}}{2}-\frac{\vartheta}{\Lambda_{N,s}}\frac{c_{N,s}}{2}\right).$$

This and (3.35) give that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_\lambda(x)-w_\lambda(y))^2}{|x-y|^{N+2s}}\,dx\,dy=0\,.$$

This implies that w_λ is constant and the claim (3.25) follows since w_λ is zero on $\{x_1=\lambda\}$.

Now we define the set

$$\Lambda:=\{\lambda\in\mathbb{R}\,:\,u\leq u_\mu\text{ in }\ \Sigma_\mu\ \forall\mu\leq\lambda\}.$$

Notice that (3.25) implies that $\Lambda\neq\emptyset$, and therefore we can consider

$$\bar\lambda:=\sup\Lambda. \tag{3.36}$$

We will show that

$$\bar\lambda=0. \tag{3.37}$$

Let us assume by contradiction that $\bar\lambda<0$. Now, in this case, we are going to show that we can move the plane a little further to the right reaching a contradiction with the definition (3.36).

First, we prove that

$$u<u_{\bar\lambda}\text{ in }\Sigma_{\bar\lambda}. \tag{3.38}$$

Indeed, by continuity, we have that $u\leq u_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$ (say outside the reflected point of the origin $0_{\bar\lambda}$). On the other hand, Lemma 3.3 implies that there exists $\rho>0$ such that

$$u<u_{\bar\lambda}\quad\text{in}\quad B_\rho(0_{\bar\lambda})\subset\Sigma_{\bar\lambda}. \tag{3.39}$$

We take $x_0\in\Sigma_{\bar\lambda}\setminus\{0_{\bar\lambda}\}$ and we fix $\bar\rho>0$ such that $B_{\bar\rho}(x_0)\subset\Sigma_{\bar\lambda}\setminus\{0_{\bar\lambda}\}$.

We set

$$\omega_{\bar\lambda}\,:=\,u_{\bar\lambda}-u.$$

Notice that $\omega_{\bar\lambda} \in \dot H^s(\mathbb{R}^N) \cap C(B_{\bar\rho}(x_0))$ and $\omega_{\bar\lambda} \geq 0$ in $\Sigma_{\bar\lambda}$. Moreover, since $|x| \geq |x_{\bar\lambda}|$ and $u \leq u_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$, using the weak formulations of (1.1) and (3.18), we have that

$$\begin{aligned}
& \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\omega_{\bar\lambda}(x)-\omega_{\bar\lambda}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
= \; & \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_{\bar\lambda}(x)-u_{\bar\lambda}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& \quad -\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
= \; & \vartheta\int_{\mathbb{R}^N}\frac{u_{\bar\lambda}(x)}{|x_{\bar\lambda}|^{2s}}\,\varphi(x)\,dx+\int_{\mathbb{R}^N}u_{\bar\lambda}^{2^*_s-1}(x)\varphi(x)\,dx \\
& \quad -\vartheta\int_{\mathbb{R}^N}\frac{u(x)}{|x|^{2s}}\,\varphi(x)\,dx-\int_{\mathbb{R}^N}u^{2^*_s-1}(x)\varphi(x)\,dx \\
\geq \; & \vartheta\int_{\mathbb{R}^N}\frac{u_{\bar\lambda}(x)-u(x)}{|x|^{2s}}\,\varphi(x)\,dx+\int_{\mathbb{R}^N}\left(u_{\bar\lambda}^{2^*_s-1}(x)-u^{2^*_s-1}(x)\right)\varphi(x)\,dx \\
\geq \; & 0,
\end{aligned}$$

for any nonnegative test function φ with compact support in $B_\rho(x_0)$. This implies that

$$(-\Delta)^s\omega_{\bar\lambda} \geq 0 \quad \text{ in } B_\rho(x_0)$$

in the weak sense. Therefore, $\omega_{\bar\lambda}$ satisfies the hypotheses of Proposition 3.1, and so either $\omega_{\bar\lambda} \equiv 0$ in $\mathbb{R}^N$ or $\omega_{\bar\lambda} > 0$ in $B_{\bar\rho}(x_0)$.

If $\omega_{\bar\lambda} \equiv 0$ in $\mathbb{R}^N$, then $u = u_{\bar\lambda}$ in $\mathbb{R}^N$, which contradicts (3.39). Therefore $\omega_{\bar\lambda} > 0$ in $B_{\bar\rho}(x_0)$, which implies that $u < u_{\bar\lambda}$ in $B_{\bar\rho}(x_0)$. Since x_0 is an arbitrary point in $\Sigma_{\bar\lambda}\backslash\{0_{\bar\lambda}\}$, this implies (3.38).

Now, notice that the inequality in (3.35) holds for any $\lambda < 0$ and the constant C_3 there is independent of λ. Moreover, since $\bar\lambda < 0$, there exists $\varepsilon_1 > 0$ such that $\bar\lambda+\varepsilon < 0$ for any $\varepsilon \in (0, \varepsilon_1)$. Recalling the notation introduced in (3.21) and (3.22), we consider the function $w_{\bar\lambda+\epsilon}$. Using the same notation as above let us consider $w_{\bar\lambda+\varepsilon}$ so that

$$supp\ w_{\bar\lambda+\varepsilon} \equiv \mathcal{S}_{\bar\lambda+\varepsilon} \cup \mathcal{D}_{\bar\lambda+\varepsilon}\,.$$

Exploiting the fact that $u < u_{\bar\lambda}$ in $\Sigma_{\bar\lambda}$ and the fact that the solution u is continuous in $\mathbb{R}^N \setminus \{0\}$ and (3.19), we deduce that:
given any $R > 0$ (large) and $\delta > 0$ (small) we can fix $\bar\varepsilon = \bar\varepsilon(R, \delta) > 0$ such that

$$\mathcal{S}_{\bar\lambda+\varepsilon} \cap \Sigma_{\bar\lambda-\delta} \subset \mathbb{R}^N \setminus B_R(0) \qquad \text{for any} \quad 0 \leq \varepsilon \leq \bar\varepsilon\,. \tag{3.40}$$

We repeat now the argument above using $w_{\bar\lambda+\varepsilon}$ as test function in the same fashion as we did using w_λ and get again

(3.41)

$$\begin{aligned}
& \left(\frac{c_{N,s}}{2}-\frac{\vartheta}{\Lambda_{N,s}}\frac{c_{N,s}}{2}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_{\bar\lambda+\varepsilon}(x)-w_{\bar\lambda+\varepsilon}(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\
\leq \; & \bar C(N,s)\left(\int_{\mathcal{S}_{\bar\lambda+\varepsilon}}u^{2^*_s}\,dx\right)^{\frac{2^*_s-2}{2^*_s}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_{\bar\lambda+\varepsilon}(x)-w_{\bar\lambda+\varepsilon}(y))^2}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned}$$

Since

$$
\begin{aligned}
\int_{\mathcal{S}_{\bar\lambda+\varepsilon}} u^{2^*_s}\,dx &\le \int_{\mathcal{S}_{\bar\lambda+\varepsilon}\cap\Sigma_{\bar\lambda-\delta}} u^{2^*_s}\,dx + \int_{\Sigma_{\bar\lambda+\varepsilon}\setminus\Sigma_{\bar\lambda-\delta}} u^{2^*_s}\,dx \\
&= \int_{\mathbb{R}^N\setminus B_R(0)} u^{2^*_s}\,dx + \int_{\Sigma_{\bar\lambda+\varepsilon}\setminus\Sigma_{\bar\lambda-\delta}} u^{2^*_s}\,dx
\end{aligned} \tag{3.42}
$$

for R large and δ small, choosing $\bar\varepsilon(R,\delta)$ as above and eventually reducing it, we can assume that

$$
\bar C(N,s)\left(\int_{\mathcal{S}_{\bar\lambda+\varepsilon}} u^{2^*_s}\,dx\right)^{\frac{2^*_s-2}{2^*_s}} < \left(\frac{c_{N,s}}{2} - \frac{\vartheta}{\Lambda_{N,s}}\frac{c_{N,s}}{2}\right),
$$

for $\bar C(N,s)$ given by (3.41). Then from (3.41) we reach that $w_{\bar\lambda+\varepsilon}=0$ and a this contradiction to (3.36). Therefore

$$
\bar\lambda = 0\,.
$$

Finally, the symmetry (and monotonicity) in the x_1-direction follows as standard repeating the argument in the $(-x_1)$-direction. The radial symmetry result (and the monotonicity of the solution) follows as well performing the *Moving Plane Method* in any direction $\nu\in S^{N-1}$. $\square$

4. Asymptotic analysis of solutions to equation (1.1)

In this section we investigate the behaviour of a solution of (1.1) near the origin and at infinity and we prove Theorem 1.7.

4.1. **A representation formula and an equivalent nonlocal problem.** In order to study the behaviour of the solutions to (1.1) near the origin and at infinity, we are going to use a representation result by Frank, Lieb and Seiringer, in particular equality (4.3) proved in [15, pag. 935]. We have the following:

Lemma 4.1. *(Ground State Representation) Let $0<\alpha<(N-2s)/2$ and let $u\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$. Set also $v_\alpha(x):=|x|^\alpha u(x)$. Then*

$$
\begin{aligned}
&\frac{1}{2}c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - (\Lambda_{N,s}+\Phi_{s,N}(\alpha))\int_{\mathbb{R}^N}|x|^{-2s}|u(x)|^2\,dx \\
&= \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\alpha(x)-v_\alpha(y)|^2}{|x-y|^{N+2s}}\frac{dx}{|x|^\alpha}\frac{dy}{|y|^\alpha},
\end{aligned} \tag{4.1}
$$

where $\Phi_{s,N}(\cdot)$ is given by

$$
\Phi_{s,N}(\alpha) = 2^{2s}\left(\frac{\Gamma(\frac{\alpha+2s}{2})\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{N-\alpha-2s}{2})\Gamma(\frac{\alpha}{2})} - \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}\right). \tag{4.2}
$$

Remark 4.2. *The result in Lemma 4.1 in particular shows that he constant $\Lambda_{N,s}$ in the Hardy-Sobolev inequality* (1.2) *is optimal and is not attained. This is the peculiar spectral behavior of the Hardy potential, motivated by the fact that the potential $|x|^{-2s}$ is in the Marcinkievicz space $\mathcal{M}^{\frac{N}{2s},\infty}$ but not in the space $L^{\frac{N}{2s}}_{loc}(\mathbb{R}^N)$. See for details Remark 4.2 in* [15].

Also, see [15, Lemma 3.2], the function $\Phi_{s,N}(\cdot)$ is negative and strictly increasing in $(0,(N-2s)/2)$ with $\Phi_{s,N}((N-2s)/2)=0$, that is

Proposition 4.3. [15, Lemma 3.2] *Consider the function*

$$\begin{array}{rcl}\Psi_{s,N}:\left[0,\dfrac{N-2s}{2}\right] & \to & [0,\Lambda_{N,s}] \\ \alpha & \to & \Psi_{s,N}(\alpha):=\Lambda_{N,s}+\Phi_{s,N}(\alpha),\end{array}$$

with $\Phi_{s,N}(\cdot)$ *defined in* (4.2). *Then* $\Psi_{s,N}$ *is strictly increasing and surjective.*

Given $\theta\in(0,\Lambda_{N,s})$ in (1.1), by Proposition 4.3, it follows the existence of a unique $\alpha\in(0,(N-2s)/2)$ such that

$$\Psi_{s,N}(\alpha)=\theta. \tag{4.3}$$

We will denote by α_θ the solution of (4.3).

Then, by (1.6) with $\varphi:=u$ and (4.1), we get

$$\begin{aligned}\int_{\mathbb{R}^N}u^{2^*_s}\,dx &= \frac{1}{2}c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy-\theta\int_{\mathbb{R}^N}|x|^{-2s}|u(x)|^2\,dx\\ &= \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\alpha(x)-v_\alpha(y)|^2}{|x-y|^{N+2s}}\frac{dx}{|x|^\alpha}\frac{dy}{|y|^\alpha}.\end{aligned} \tag{4.4}$$

On the other hand, recalling that

$$v_\alpha(x):=|x|^\alpha u(x), \tag{4.5}$$

with $\alpha=\alpha_\theta$, from (4.4) we get

$$\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\alpha(x)-v_\alpha(y)|^2}{|x-y|^{N+2s}}\frac{dx}{|x|^\alpha}\frac{dy}{|y|^\alpha}=\int_{\mathbb{R}^N}\frac{v_\alpha^{2^*_s}(x)}{|x|^{\alpha\cdot 2^*_s}}\,dx. \tag{4.6}$$

This suggests to define the space $\dot{H}^{s,\alpha}(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|\phi\|_{\dot{H}^{s,\alpha}(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N}\frac{|\phi(x)|^{2^*_s}}{|x|^{\alpha 2^*_s}}\,dx\right)^{\frac{1}{2^*_s}}+\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\frac{dx}{|x|^\alpha}\frac{dy}{|y|^\alpha}\right)^{\frac{1}{2}}.$$

We also define

$$\dot{W}^{s,\alpha}(\mathbb{R}^N):=\{\phi:\mathbb{R}^N\to\mathbb{R}\text{ measurable }:\|\phi\|_{\dot{H}^{s,\alpha}(\mathbb{R}^N)}<+\infty\}.$$

Note that, following e.g. [18], one has that the space $\dot{W}^{s,\alpha}(\mathbb{R}^N)$ coincides with $\dot{H}^{s,\alpha}(\mathbb{R}^N)$.

Remark 4.4. *Notice that Lemma 4.1 says that equality* (4.1) *holds true for functions* $u\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$. *Actually, by a density argument one can prove that it holds for any* $u\in\dot{H}^s(\mathbb{R}^N)$.

Indeed, one first approximates a function $u\in\dot{H}^s(\mathbb{R}^N)$ *with a function* $u_\epsilon\in C_0^\infty(\mathbb{R}^N)$. *Then, one uses a standard cut-off argument near zero. In this way, one can see that in the left hand side of* (4.1) *it is possible to pass to the limit.*

In order to pass to the limit also in the right hand side of (4.1), *one needs to notice that, by the representation formula, Cauchy sequences are sent into Cauchy sequences and we are working in Hilbert spaces. Therefore, the conclusion follows by observing that on the left hand side we have a Cauchy sequence since it is convergent.*

As a consequence of the *ground state representation* given by Lemma 4.1, we will transform our problem (1.1) into another nonlocal problem in weighted spaces. Namely, we consider $u \in \dot{H}^s(\mathbb{R}^N)$ that is a solution of the problem

$$(-\Delta)^s u = \vartheta \frac{u}{|x|^{2s}} + u^{2_s^*-1} \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

and we set $v_\alpha(x) := |x|^\alpha u(x)$ with $\alpha = \alpha_\theta$ given by (4.3).

By Lemma 4.1, Remark 4.4 and [1], it follows that $v_\alpha \in \dot{H}^{s,\alpha}(\mathbb{R}^N)$.

Furthermore, we introduce the operator $(-\Delta_\alpha)^s$, defined as duality product

$$< (-\Delta_\alpha)^s v, \phi >= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \frac{dx}{|x|^\alpha} \frac{dy}{|y|^\alpha}, \tag{4.7}$$

for any $\phi \in \dot{H}^{s,\alpha}(\mathbb{R}^N)$. With this notation, we have that v_α is a weak solution to

$$(-\Delta_\alpha)^s v = \frac{v^{2_s^*-1}}{|x|^{\alpha \cdot 2_s^*}} \quad \text{in } \mathbb{R}^N, \tag{4.8}$$

namely for any $\phi \in \dot{H}^{s,\alpha}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \frac{dx}{|x|^\alpha} \frac{dy}{|y|^\alpha} = \int_{\mathbb{R}^N} \frac{v^{2_s^*-1}(x)}{|x|^{\alpha \cdot 2_s^*}} \phi(x)\, dx.$$

Summarizing, with the *ground state representation* we have hidden the Hardy potential and the cost is that we now have to handle $(-\Delta_\alpha)^s$, that is an elliptic operator with singular coefficients.

On the other hand, to get the exact behavior of the solution u to (1.1) near the origin and at infinity, it is sufficient to get an L^∞ estimate and the Harnack inequality for v_α. This is the main goal of the forthcoming Subsections 4.2 and 4.3.

4.2. **A regularity result: the L^∞ estimate.** In this section we prove a regularity result for weak solution of (4.8). More precisely:

Proposition 4.5. *Let $\alpha \in (0, (N-2s)/2)$. Let $v \in \dot{H}^{s,\alpha}(\mathbb{R}^N)$ be a nonnegative weak solution of*

$$(-\Delta_\alpha)^s v = \frac{v^{2_s^*-1}}{|x|^{\alpha \cdot 2_s^*}} \quad \textit{in } \mathbb{R}^N.$$

Then $v \in L^\infty(\mathbb{R}^N)$.

Proof. Let us define for $\beta \geq 1$ and $T > 0$

$$\phi(t) = \phi_{\beta,T}(t) = \begin{cases} t^\beta, & \text{if } \ 0 \leq t \leq T \\ \beta T^{\beta-1}(t - T) + T^\beta, & \text{if } \ t > T. \end{cases}$$

We observe that (as in the case of the standard fractional laplacian $(-\Delta)^s(\cdot)$, see [28]) it holds in the weak distributional meaning that

$$(-\Delta_\alpha)^s \phi(v) \leq \phi'(v)(-\Delta_\alpha)^s v, \ v \in \dot{H}^{s,\alpha}(\mathbb{R}^N). \tag{4.9}$$

Since $\phi_{\beta,T}$ is a Lipschitz function it follows that $\phi_{\beta,T}(v) \in \dot{H}^{s,\alpha}(\mathbb{R}^N)$. By using the weighted Sobolev inequality we have

$$\begin{aligned}(4.10)\qquad & \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(v(x))-\phi(v(y))|^2}{|x-y|^{N+2s}}\frac{dx}{|x|^\alpha}\frac{dy}{|y|^\alpha} \\ \geq\ & \frac{c_{N,s}}{2}S(N,s,\alpha)\left(\int_{\mathbb{R}^N}|\phi(v)|^{2_s^*}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{2}{2_s^*}}.\end{aligned}$$

On the other hand, using (4.9), we get

$$\begin{aligned}(4.11)\qquad & \int_{\mathbb{R}^N}\phi(v)(-\Delta_\alpha)^s\phi(v) \leq \int_{\mathbb{R}^N}\phi(v)\phi'(v)(-\Delta_\alpha)^s v \\ =\ & \int_{\mathbb{R}^N}\phi(v)\phi'(v)v^{2_s^*-1}\frac{dx}{|x|^{\alpha\cdot 2_s^*}} \leq \beta\int_{\mathbb{R}^N}(\phi(v))^2 v^{2_s^*-2}\frac{dx}{|x|^{\alpha\cdot 2_s^*}},\end{aligned}$$

where we used that $t\phi'(t)\leq \beta\phi(t)$. From (4.10) and (4.11) we obtain

$$(4.12)\qquad \left(\int_{\mathbb{R}^N}|\phi(v)|^{2_s^*}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{2}{2_s^*}} \leq C\beta\int_{\mathbb{R}^N}(\phi(v))^2 v^{2_s^*-2}\frac{dx}{|x|^{\alpha\cdot 2_s^*}},$$

for some positive constant C. Now, in order to apply the Moser's iteration technique in $\mathbb{R}^N$ and get then the local boundedness of the solution, we take into account that

$$(4.13)\qquad \int_{\mathbb{R}^N}\frac{v^{2_s^*}}{|x|^{\alpha\cdot 2_s^*}} < +\infty$$

and we estimate the right hand side of (4.12). Let

$$(4.14)\qquad \beta = \frac{2_s^*}{2}$$

and let $m\in\mathbb{R}^+$ to be chosen later. We have

$$\begin{aligned}(4.15)\qquad & \\ & C\beta\int_{\mathbb{R}^N}(\phi(v))^2 v^{2_s^*-2}\frac{dx}{|x|^{\alpha\cdot 2_s^*}} \\ =\ & C\beta\int_{\{v\leq m\}\cap\mathbb{R}^N}(\phi(v))^2 v^{2_s^*-2}\frac{dx}{|x|^{\alpha\cdot 2_s^*}} + C\beta\int_{\{v\geq m\}\cap\mathbb{R}^N}\frac{(\phi(v))^2}{|x|^{2\alpha}}\frac{v^{2_s^*-2}}{|x|^{\alpha\cdot(2_s^*-2)}}\,dx \\ \leq\ & C\beta\int_{\{v\leq m\}\cap\mathbb{R}^N}(\phi(v))^2 m^{2_s^*-2}\frac{dx}{|x|^{\alpha\cdot 2_s^*}} + C\beta\left(\int_{\{v\geq m\}\cap\mathbb{R}^N}\frac{(\phi(v))^{2_s^*}}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{2}{2_s^*}}\left(\int_{\{v\geq m\}\cap\mathbb{R}^N}\frac{v^{2_s^*}}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{2_s^*-2}{2_s^*}},\end{aligned}$$

where in the last term we used Hölder inequality with exponents $2_s^*/2$ and $2_s^*/(2_s^*-2)$. Since (4.13) holds, we can fix m such that

$$\left(\int_{\{v\geq m\}\cap\mathbb{R}^N}\frac{v^{2_s^*}}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2C\beta},$$

and then from (4.12) and (4.15) we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|\phi(v)|^{2_s^*}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{2}{2_s^*}} &\leq C\beta m^{2_s^*-2}\int_{\mathbb{R}^N}(\phi(v))^2\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\\ &\leq C\beta m^{2_s^*-2}\int_{\mathbb{R}^N}v^{2\beta}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}<+\infty,\end{aligned}$$

where we used that $\phi(t)\leq t^\beta$, (4.13) and (4.14). By Fatou's lemma, taking $T\to+\infty$ one has

$$\left(\int_{\mathbb{R}^N}|v|^{\beta\cdot 2_s^*}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{2}{2_s^*}}<+\infty. \tag{4.16}$$

The result now, follows using Moser's iteration, see e.g. [28, Theorem 13]. For $k\geq 1$, let us define $\{\beta_k\}$ by

$$2\beta_{k+1}+2_s^*-2=2_s^*\beta_k \quad \text{and} \quad \beta_1=\frac{2_s^*}{2}.$$

Then from (4.12) and using (4.16), iterating we obtain

$$\left(\int_{\mathbb{R}^N}|v|^{\beta_{k+1}\cdot 2_s^*}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{1}{2_s^*(\beta_{k+1}-1)}}\leq\left(C\beta_{k+1}\right)^{\frac{1}{2(\beta_{k+1}-1)}}\left(\int_{\mathbb{R}^N}v^{\beta_k\cdot 2_s^*}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{1}{2_s^*(\beta_k-1)}}.$$

If we denote

$$A_k:=\left(\int_{\mathbb{R}^N}|v|^{\beta_k\cdot 2_s^*}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{1}{2_s^*(\beta_k-1)}}, \qquad C_k:=(C\beta_k)^{\frac{1}{2(\beta_k-1)}},$$

we get the recurrence formula $A_{k+1}\leq C_{k+1}A_k$, $k\geq 1$. Arguing by induction we have

$$\begin{aligned}\log A_{k+1} &\leq \sum_{j=2}^{k+1}\log C_j+\log A_1\\ &\leq \sum_{j=2}^{+\infty}\log C_j+\log A_1<+\infty,\end{aligned} \tag{4.17}$$

since the serie $\sum_{j=2}^{+\infty}\log C_j<+\infty$ is convergent (recall that $\beta_{k+1}=\beta_1^k(\beta_1-1)+1$) and $A_1\leq C$, see (4.16). For $R>0$ fixed, using (4.17), it follows

$$\log\left(\left(\int_{B_R}|v|^{\beta_{k+1}\cdot 2_s^*}\frac{dx}{|x|^{\alpha\cdot 2_s^*}}\right)^{\frac{1}{2_s^*(\beta_{k+1}-1)}}\right)\leq C$$

and then

$$\frac{\alpha}{(\beta_{k+1}-1)}\log\frac{1}{R}+\log\left(\left(\int_{B_R}|v|^{\beta_{k+1}\cdot 2_s^*}\,dx\right)^{\frac{1}{2_s^*(\beta_{k+1}-1)}}\right)\leq C.$$

Since $\beta_k\to+\infty$ as $k\to\infty$, we have

$$\log\left(\left(\int_{B_R}|v|^{\beta_{k+1}\cdot 2_s^*}\,dx\right)^{\frac{1}{2_s^*(\beta_{k+1}-1)}}\right)\leq C,$$

with C a positive constant not depending on R. This end the proof since

$$\lim_{k\to+\infty}\left(\int_{B_R}|v|^{\beta_{k+1}\cdot 2_s^*}\,dx\right)^{\frac{1}{2_s^*\beta_{k+1}}}=\|v\|_{L^\infty(B_R)}$$

and thus

$$\|v\|_{L^\infty(\mathbb{R}^N)} \le C,$$

which gives the desired result. □

4.3. **Harnack inequality for v_α.** A good reference in order to understand the differences between the Harnack inequality for local and nonlocal operators related to the fractional Laplacian is the paper [24]. However, such a paper does not apply directly to the operator $(-\Delta_\alpha)^s$ introduced in (4.7), because of the singularities of the coefficients of the operator.

For our purposes we are going to use the following weak Harnack inequality, that has been obtained in [2].

Theorem 4.6. *Let $\alpha \in (0,(N-2s)/2)$. Let $v \in \dot{H}^{s,\alpha}(\mathbb{R}^N)$ be a nonnegative solution of*

$$(-\Delta_\alpha)^s v = \frac{v^{2_s^*-1}}{|x|^{\alpha \cdot 2_s^*}} \quad \text{in } \mathbb{R}^N. \tag{4.18}$$

Then, for $1 \le q < \frac{N}{N-2s}$ the following inequality holds true

$$\Big(\int_{B_r} v^q d\mu(x)\Big)^{\frac{1}{q}} \le C(q,N,s) \inf_{B_{\frac{3}{2}r}} v, \quad d\mu(x) := \frac{dx}{|x|^{2\alpha}}. \tag{4.19}$$

For the readers convenience we describe the strategy of the proof in a schematic way and we refer to [2] for the details. The functional framework needed to prove Theorem 4.6 can be found in Appendix B of [3], where a Harnack parabolic inequality is obtained for the heat equation corresponding to the elliptic operator $(-\Delta_\alpha)^s$.

The proof of Theorem 4.6 is carried out using classical arguments by Moser and by Krylov-Safonov.

In the local case (that is, when $s = 1$ and the fractional Laplacian reduces to the Laplacian), the Harnack inequality for elliptic operator with weights has been proved in [12]. In the nonlocal case, we also refer to the paper [11], in which the authors consider nonlinear operators of nonlocal p-Laplacian type. We also refer to Chapter 7 of the book of Giusti [20].

For simplicity of notation, we will write B_r in place of $B_r(0)$. Moreover, we will use the notation

$$d\mu := \frac{dx}{|x|^{2\alpha}} \quad \text{and} \quad d\nu := \frac{dx\,dy}{|x-y|^{N+2s}|x|^\alpha|y|^\alpha}.$$

The first result toward the proof of the Harnack inequality is contained in the following lemma, where we check that, even in the presence of singular weights, we get a *propagation of positivity* result. More precisely:

Lemma 4.7. *(Propagation of positivity) Assume that $v \in \dot{H}^{s,\alpha}(\mathbb{R}^N)$, with $v \gneqq 0$ in $B_R(0)$ with $0 < R < 1$, is a supersolution to (4.18). Let $k > 0$ and suppose that for some $\sigma \in (0,1]$, we have*

$$|B_r \cap \{v \ge k\}|_{d\mu} \ge \sigma |B_r|_{d\mu} \tag{4.20}$$

with $0 < r < \frac{R}{16}$, then there exists a positive constant $C = C(N,s)$ such that

$$|B_{6r} \cap \{v \le 2\delta k\}|_{d\mu} \le \frac{C}{\sigma \log(\frac{1}{2\delta})} |B_{6r}|_{d\mu} \tag{4.21}$$

for all $\delta \in (0,\frac{1}{4})$.

We also mention the paper [19], that contains some estimates that are useful to handle radial weights.

An iterative argument as in Lemma 3.2 in [11] gives the following local estimate on the infimum of v.

Lemma 4.8. *Assume that the hypotheses of Lemma 4.7 are satisfied. Then there exists* $\delta \in (0, \frac{1}{2})$ *such that*

$$\inf_{B_{4r}} v \geq \delta k. \tag{4.22}$$

Since the weight $|x|^{-2\alpha}$ is an *admissible weight* in the sense defined in [22], we obtain the following *reverse Hölder inequality* for v.

Lemma 4.9. *(Reverse Hölder inequality) Suppose that v is a nonnegative supersolution to* (4.18), *then for all* $0 < \gamma_1 < \gamma_2 < \frac{N}{N-2s}$, *we have*

$$\left(\frac{1}{|B_r|_{d\mu}} \int\limits_{B_r} v^{\gamma_2} \, d\mu \right)^{\frac{1}{\gamma_2}} \leq C \left(\frac{1}{|B_{\frac{3}{2}r}|_{d\mu}} \int\limits_{B_{\frac{3}{2}r}} v^{\gamma_1} \, d\mu \right)^{\frac{1}{\gamma_1}}. \tag{4.23}$$

In order to prove the main result of this section, we also need to estimate an average of v by the infimum in a small ball. For this, we state the following covering lemma in the spirit of Krylov-Safonov theory (see [25] for a proof in a very general framework).

Lemma 4.10. *Assume that $E \subset B_r(x_0)$ is a measurable set. For $\bar\delta \in (0,1)$, we define*

$$\big[E\big]_{\bar\delta} := \bigcup_{\rho>0} \{B_{3\rho}(x) \cap B_r(x_0), x \in B_r(x_0) : |E \cap B_{3\rho}(x)|_{d\mu} > \bar\delta |B_\rho(x)|_{d\mu}\}.$$

Then, there exists $\tilde C$ depending only on N, such that, either

(1) $|\big[E\big]_{\bar\delta}|_{d\mu} \geq \dfrac{\tilde C}{\bar\delta} |E|_{d\mu}$, *or*
(2) $\big[E\big]_{\bar\delta} = B_r(x_0)$.

With this, we can have the following:

Lemma 4.11. *(Main result) Assume that v is a nonnegative supersolution to* (4.18), *then there exists $\eta \in (0,1)$ depending only on N, s such that*

$$\Big(\frac{1}{|B_r|_{d\mu}} \int_{B_r} v^\eta d\mu(x) \Big)^{\frac{1}{\eta}} \leq C \inf_{B_r} v. \tag{4.24}$$

Proof. For any $\eta > 0$ we have,

$$\frac{1}{|B_r|_{d\mu}} \int_{B_r} v^\eta d\mu(x) = \eta \int_0^\infty t^{\eta-1} \frac{|B_r \cap \{v > t\}|_{d\mu}}{|B_r|_{d\mu}} dt. \tag{4.25}$$

For any $t > 0$ and $i \in \mathbb{N}$, we set $A_t^i := \{x \in B_r : v(x) > t\delta^i\}$, where δ is given by Lemma 4.8. It is easy to check that $A_t^{i-1} \subset A_t^i$.

Let $x \in B_r$ such that $B_{3\rho}(0) \cap B_r \subset [A_t^{i-1}]_{\bar{\delta}}$, then

$$|A_t^{i-1} \cap B_{3\rho}(x)|_{d\mu} > \bar{\delta}|B_\rho|_{d\mu} = \frac{\bar{\delta}}{3^{N-2\gamma}}|B_{3\rho}|_{d\mu}.$$

Hence, using Lemma 4.8, we obtain that

$$v(x) > \delta(t\delta^{i-1}) = t\delta^i \quad \text{ for all } x \in B_r.$$

Thus $[A_t^{i-1}]_{\bar{\delta}} \subset A_t^i$. Therefore, using the alternatives in Lemma 4.10, we obtain that

- either $A_t^i = B_r$
- or $|A_t^i|_{d\mu} \geq \dfrac{\tilde{C}}{\delta}|A_t^{i-1}|_{d\mu}$.

Hence, if for some $m \in \mathbb{N}$ we have

$$|A_t^0|_{d\mu} > \left(\frac{\bar{\delta}}{\tilde{C}}\right)^m |B_r|_{d\mu}, \tag{4.26}$$

then $|A_t^m|_{d\mu} = |B_r|_{d\mu}$. Therefore $A_t^i = B_r$ and then

$$\inf_{B_r} v > t\delta^m.$$

It is clear that (4.26) holds if $m > \frac{1}{\log\left(\frac{\bar{\delta}}{\tilde{C}}\right)} \log\left(\dfrac{|A_t^0|_{d\mu}}{|B_r|_{d\mu}}\right)$.

Fix m as above and define

$$\beta := \frac{\log\left(\dfrac{\bar{\delta}}{\tilde{C}}\right)}{\log(\delta)}.$$

It follows that

$$\inf_{B_r} v > t\delta\left(\frac{|A_t^0|_{d\mu}}{|B_r|_{d\mu}}\right)^{\frac{1}{\beta}}.$$

We set $\xi := \inf_{B_r} v$, then

$$\frac{|B_r \cap \{v > t\}|_{d\mu}}{|B_r|_{d\mu}} = \frac{|A_t^0|_{d\mu}}{|B_r|_{d\mu}} \leq C\delta^{-\beta}t^{-\beta}\xi^\beta.$$

Going back to (4.25), we have

$$\frac{1}{|B_r|_{d\mu}}\int_{B_r} v^\eta d\mu(x) \leq \eta\int_0^a t^{\eta-1}dt + \eta C\int_a^\infty \delta^{-\beta}t^{-\beta}\xi^\beta dt.$$

Choosing $a = \xi$ and $\eta = \frac{\beta}{2}$, we obtain the desired result. $\square$

Proof of Theorem 4.6. Using Lemma 4.11 we obtain that

$$\left(\frac{1}{|B_r|_{d\mu}}\int_{B_r} u^\eta d\mu(x)\right)^{\frac{1}{\eta}} \leq C \inf_{B_r} u$$

for some $\eta \in (0, 1)$. Fixed $1 \leq q < \frac{N}{N-2s}$, and applying Lemma 4.9 with $\gamma_1 := \eta$ and $\gamma_2 := q$, there results that

$$\left(\frac{1}{|B_r|_{d\mu}}\int_{B_r} u^q \, d\mu\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|B_{\frac{3}{2}r}|_{d\mu}}\int_{B_{\frac{3}{2}r}} u^\eta \, d\mu\right)^{\frac{1}{\eta}}. \tag{4.27}$$

Hence

$$\left(\frac{1}{|B_r|_{d\mu}}\int_{B_r} u^q\, d\mu\right)^{\frac{1}{q}} \leq C \inf_{B_{\frac{3}{2}r}} u,$$

which concludes the proof of Theorem 4.6. □

We refer to [2] for all the technical details of the proofs above and to Appendix B in [3] for the functional inequalities for weighted fractional Sobolev spaces needed in the proofs of the previous lemmata.

Remark 4.12. *With the L^∞ estimate and the weak Harnack inequality we could obtain the classical Harnack inequality. We omit the details because they are quite classical.*

4.4. **Proof of Theorem 1.7.** We start studying the behavior of the solution u near the origin. Defining

$$v_{\alpha_\theta}(x) = |x|^{\alpha_\theta} u(x),$$

for $R_0 > 0$, by the Harnack inequality, see Theorem 4.6, we get that

$$C_H \inf_{B_{R_0}} v_{\alpha_\theta}(x) \geq \left(\int_{B_{R_0}} v^q\, dx\right)^{\frac{1}{q}} \geq c_0, \tag{4.28}$$

for some positive constant c_0. On the other hand, Proposition 4.5 implies the existence of a positive constant C_0 such that

$$v_{\alpha_\theta}(x) \leq C_0, \quad x \in B_{R_0}. \tag{4.29}$$

Then, from (4.28) and (4.29) (recalling that $v_{\alpha_\theta}(x) = |x|^{\alpha_\theta} u(x)$), it follows

$$\frac{c_0}{|x|^{\alpha_\theta}} \leq u(x) \leq \frac{C_0}{|x|^{\alpha_\theta}} \quad \text{in } B_{R_0}.$$

Thus, recalling (1.12),

$$\frac{c_0}{|x|^{(1-\eta_\theta)\frac{N-2s}{2}}} \leq u(x) \leq \frac{C_0}{|x|^{(1-\eta_\theta)\frac{N-2s}{2}}} \quad \text{in } B_{R_0}. \tag{4.30}$$

In order to study the behavior of $u(x)$ at infinity, we use the Fractional Kelvin transform, see e.g. [31, Proposition A.1]. Let $x \to x^* = x/|x|^2$ the inversion with respect to the unit sphere and let us define

$$u^*(x) := |x|^{2s-N} u(x^*). \tag{4.31}$$

Then, from [31, Proposition A.1] we have that

$$(-\Delta)^s u^*(x) = \frac{1}{|x|^{N+2s}} (-\Delta)^s u(x^*), \quad x \neq 0. \tag{4.32}$$

Using (1.1) and (4.32), formally we obtain that

$$\begin{aligned}(-\Delta)^s u^*(x) &= \frac{1}{|x|^{N+2s}}\left[\theta\frac{u\left(\frac{x}{|x|^2}\right)}{\left|\frac{x}{|x|^2}\right|^{2s}} + u^{2^*_s-1}\left(\frac{x}{|x|^2}\right)\right]\\ &= \theta\frac{u^*(x)}{|x|^{2s}} + \left(u^*\right)^{2^*_s-1}, \quad x\neq 0.\end{aligned}$$

Moreover, from [13, Lemma 2.2 and Corollary 2.3] we have that $u^*\in\dot{H}^s(\mathbb{R}^N)$ and it is a weak solution of the problem

$$(-\Delta)^s u^*(x) = \theta\frac{u^*(x)}{|x|^{2s}} + \left(u^*\right)^{2^*_s-1} \quad \text{in } \mathbb{R}^N\setminus\{0\}. \tag{4.33}$$

Then arguing as in the first part of the proof, for a fixed $1/R_\infty > 0$, there exist two positive constants c_∞ and C_∞ such that

$$\frac{c_\infty}{|x|^{\alpha_\theta}} \le u^*(x) \le \frac{C_\infty}{|x|^{\alpha_\theta}} \quad \text{in } B_{\frac{1}{R_\infty}}. \tag{4.34}$$

Scaling back in (4.34), see (4.31), we obtain

$$\frac{c_\infty}{|x|^{(1+\eta_\theta)\frac{N-2s}{2}}} \le u(x) \le \frac{C_\infty}{|x|^{(1+\eta_\theta)\frac{N-2s}{2}}} \quad \text{in } \mathbb{R}^N\setminus B_{R_\infty}. \tag{4.35}$$

Redefining constants, from (4.30) and (4.35), we get (1.11), and this concludes the proof of Theorem 4.6.

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